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# A Class of Quadrature Formulas of Chebyshev Type for Singular Integrals

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## 1. INTRODUCTION

The extension of quadrature formulas of Chebyshev type to singular integrals had been investigated by several authors [2–4, 7]. The Gauss–Chebyshev method of numerical integration was extended to singular integrals by Erdogen and Gupta [3]. Let

$$I(x) = I(x, g) = \frac{1}{\pi} \int_{-1}^{1} \frac{g(t)}{t - x} \frac{dt}{\sqrt{1 - t^2}}, \qquad -1 < x < 1; \tag{1.1}$$

$$J(t) = J(t, g) = \frac{1}{\pi} \int_{-1}^{1} \frac{g(x)}{x - t} \sqrt{1 - x^2}\, dx, \qquad -1 \leqslant t \leqslant 1. \tag{1.2}$$

They obtained the approximate formulas

$$I(x_j) \approx \frac{1}{n} \sum_{k=1}^{n} \frac{g(t_k)}{t_k - x_j}, \qquad j = 1, \ldots, n-1, \tag{1.3}$$

$$J(t_k) \approx \frac{1}{n} \sum_{j=1}^{n-1} \frac{g(x_j)(1 - x_j^2)}{x_j - t_k}, \qquad k = 1, \ldots, n, \tag{1.4}$$

where

$$t_k = \cos \frac{2k-1}{2n} \pi, \qquad x_j = \cos \frac{j}{n} \pi \tag{1.5}$$

are the zeros of Chebyshev polynomials $T_n(x)$ of the first kind of degree $n$ and $U_{n-1}(x)$ of the second kind of degree $n-1$, respectively:

$$T_n(x) = \cos n\theta, \qquad U_{n-1}(x) = \sin n\theta / \sin \theta, \qquad x = \cos \theta. \tag{1.6}$$



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Chawla and Ramakrishnan [2] extended (1.3) and (1.4) to the forms

$$I(x) \approx \frac{1}{n} \sum_{k=1}^{n} \frac{g(t_k)}{t_k - x} + g(x) \frac{U_{n-1}(x)}{T_n(x)}, \qquad -1 < x < 1, \quad x \neq t_k, \tag{1.7}$$

$$J(t) \approx \frac{1}{n} \sum_{j=1}^{n-1} \frac{g(x_j)(1 - x_j^2)}{x_j - t} - g(t) \frac{T_n(t)}{U_{n-1}(t)}, \qquad -1 \leqslant t \leqslant 1, \quad t \neq x_j, \tag{1.8}$$

respectively, under certain assumptions of analyticity of $g(x)$, gave some complicated formulas of their remainders and pointed out that they are exact for $g(x) \in \pi_{2n-1}$ and $\pi_{2n-2}$, respectively ($\pi_m$ being the class of polynomials of degree not greater than $m$).

In 1977, the Lobatto–Chebyshev (or trapezoidal Chebyshev) method of numerical intergration was extended to evaluate singular integrals by Theocaris and Ioakimidis [7]. They obtained the formula

$$I(t_k) \approx \frac{1}{n} \sum_{j=0}^{n} \lambda_j \frac{g(x_j)}{x_j - t_k}, \qquad k = 1,..., n \tag{1.9}$$

$$(x_0 = 1, x_n = -1, \lambda_0 = \lambda_n = \tfrac{1}{2}, \lambda_1 = \cdots = \lambda_{n-1} = 1),$$

and pointed out that it is exact for $g(x) \in \pi_{2n}$.

The authors of [3] and [7] derived the mentioned formulas for the purpose of solving singular integral equations of the first kind

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\varphi(x)}{x - t} dx + \int_{-1}^{1} K(x, t) \varphi(x) dx = f(t), \qquad -1 < t < 1, \tag{1.10}$$

numerically by the method of collocation. As pointed out in [7], on this purpose, (1.9) has the advantage over (1.3) in that the values of $g(\pm 1)$ may be obtained directly without any complementary procedure such as extrapolation which may give rise to significant errors, while the determination of such values is usually important in practice.

In [2], an approximate formula for the singular integral

$$H(x) = H(x, g) = \frac{1}{\pi} \int_{-1}^{1} \frac{g(t)}{t - x} \sqrt{\frac{1 + t}{1 - t}} dt, \qquad -1 \leqslant x < 1, \tag{1.11}$$

was obtained by using Jacobi polynomials $P_n^{(-1/2, 1/2)}(x)$ under similar assumptions of analyticity of $g(t)$. In [4], Erdogen *et al.* obtained the approximate formula

$$H(\sigma_k) \approx \frac{2}{2n + 1} \sum_{j=1}^{n} \frac{g(\tau_j)(1 + \tau_j)}{\tau_j - \sigma_k}, \qquad k = 1,..., n, \tag{1.12}$$

where

$$\tau_k = \cos \frac{2k-1}{2n+1}\pi, \qquad \sigma_k = \cos \frac{2k}{2n+1}\pi; \tag{1.13}$$

$(\tau_k, \sigma_k)$ actually are the zeros of $U_{2n}(x)$.

The methods used in [3], [4] and [7] were based on expanding $g(x)$ into Chebyshev series, from which it is hard to estimate the remainders. In [2], the method of complex integration was used due to the assumption of analyticity of $g(t)$ and the formula of estimation for the remainder seems inconvenient to use in applications. In this paper, we shall reprove and extend these formulas systematically by a unified method based on the corresponding formulas for ordinary integrals so that their remainders can be easily estimated. Using the same method, we also establish another set of formulas named Simpson–Chebyshev type which seems more effective when the density function $g(x)$ has less order of smoothness as illustrated by an example at the end of this paper.

Throughout the paper, we assume $g(x)$ to be smooth to certain order which is obvious from the context.

## 2. Some Lemmas

The following lemma will be applied repeatedly.

Lemma 1. *If $g(x) \in C^{n+1}[a, b]$ and $a \leqslant x_0 \leqslant b$, let*

$$G(x) = \begin{cases} \dfrac{g(x) - g(x_0)}{x - x_0}, & \text{when } x \neq x_0, \\ g'(x_0), & \text{when } x = x_0, \end{cases}$$

*then*

$$G^{(k)}(x) = \begin{cases} \dfrac{g^{(k+1)}(\xi_k)}{k+1}, & \text{when } x \neq x_0, \\ \dfrac{g^{(k+1)}(x_0)}{k+1}, & \text{when } x = x_0, \end{cases} \qquad k = 0, 1, \ldots, n, \tag{2.1}$$

*where $\xi_k$ is a value between $x$ and $x_0$.*

*Proof.* For $x \neq x_0$, it is easy to prove by induction

$$G^{(k)}(x) = k![g(x_0) - g(x) - g'(x)(x_0 - x) - \cdots - (1/k!)\, g^{(k)}(x)(x_0 - x)^k]/(x_0 - x)^{k+1}. \tag{2.2}$$

Equation (2.1) follows immediately from (2.2) by Taylor's theorem, provided $x \neq x_0$. Letting $x \to x_0$ we get (2.1) for $x = x_0$.

Another lemma for integrals of Cauchy principal value is useful.

LEMMA 2. *The following formulas are valid*:

$$\frac{1}{\pi} \int_{-1}^{1} \frac{dx}{(x-t)\sqrt{1-x^2}} = 0, \tag{2.3}$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{x-t} dx = -t, \tag{2.4}$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{1}{x-t} \sqrt{\frac{1+x}{1-x}}\, dx = 1. \tag{2.5}$$

*Proof.* By the residue theorem, we may easily verify

$$\frac{1}{2\pi i} \int_{\gamma^-} \frac{d\zeta}{(\zeta - z)\sqrt{1-\zeta^2}} = \frac{1}{\sqrt{1-z^2}},$$

where $\gamma$ is a contour surronding $[-1, 1]$ and $z$ is exterior to it, $\sqrt{1-\zeta^2}$ being the analytic continuation of $\sqrt{1-x^2}$ along the upper side of $[-1, 1]$. Let $\gamma$ shrink to $[-1, 1]$, and we get

$$\frac{1}{\pi i} \int_{-1}^{1} \frac{dx}{(x-z)\sqrt{1-x^2}} = \frac{1}{\sqrt{1-z^2}}, \qquad z \notin [-1, 1].$$

Applying Plemelj's formulas to it [6], we obtain (2.3).

Equations (2.4) and (2.5) may be obtained in a similar way.

## 3. FORMULAS OF GAUSS–CHEBYSHEV TYPE

This method is based on the formula [1]

$$I = I(f) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt \approx \frac{1}{n} \sum_{k=1}^{n} f(t_k), \tag{3.1}$$

with remainder

$$R_n[I] = \frac{1}{(2n)!\,2^{2n-1}} f^{(2n)}(\xi), \qquad -1 < \xi < 1. \tag{3.2}$$

If we put $t = \cos\theta$ in (3.1), we see that it is the same as the formula by the tangential method [5].

By (2.3), we may write (1.1) as

$$I(x, g)=\frac{1}{\pi}\int_{-1}^{1}\frac{g(t)-g(x)}{t-x}\frac{dt}{\sqrt{1-t^2}} \tag{1.1$'$}$$

which is an ordinary integral. Hence by (3.1) we have

$$I(x)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)}{t_k-x}+\frac{g(x)}{n}\sum_{k=1}^{n}\frac{1}{x-t_k}, \qquad x\neq t_k. \tag{3.3}$$

But it is evident

$$\sum_{k=1}^{n}\frac{1}{x-t_k}=\frac{T_n'(x)}{T_n(x)}=n\frac{U_{n-1}(x)}{T_n(x)} \tag{3.4}$$

since

$$T_n'(x)=nU_{n-1}(x). \tag{3.5}$$

Substituting (3.4) into (3.3), we obtain (1.7).

By Lemma 1 and (3.2), we have the remainder of (1.7)

$$R_n[I(x)]=\frac{1}{(2n)!\,2^{2n-1}}G^{(2n)}(\xi)=\frac{1}{(2n+1)!\,2^{2n-1}}g^{(2n+1)}(\xi'),$$

or its estimate

$$|R_n[I(x)]|\leqslant\frac{1}{(2n+1)!\,2^{2n-1}}M_{2n+1}(g), \tag{3.6}$$

where $M_m(g)=\max_{|x|\leqslant 1}|g^{(m)}(x)|$. Then obviously (1.7) is exact for $g(x)\in\pi_{2n}$.

In order to get $I(t_r, g)$, we note that $I(x)$ is continuous in $-1<x<1$. Therefore,

$$\begin{aligned} I(t_r)\approx{}&\frac{1}{n}\sum_{k=1}^{n}{}'\frac{g(t_k)}{t_k-t_r}+\frac{1}{n}g'(t_r)\\ &+\frac{g(t_r)}{n}\lim_{x\to t_r}\frac{nU_{n-1}(x)(x-t_r)-T_n(x)}{(x-t_r)\,T_n(x)}, \end{aligned}$$

where $\sum'$ denotes the summation except $k=r$.

Noting that

$$(1-x^2)\,U_{n-1}'(x)=U_{n-2}(x)-(n-1)\,T_n(x), \qquad \frac{U_{n-1}'(t_k)}{U_{n-1}(t_k)}=\frac{t_k}{1-t_k^2}, \tag{3.7}$$

we have

$$\lim_{x\to t_r}\frac{nU_{n-1}(x)(x-t_r)-T_n(x)}{(x-t_r)\,T_n(x)}=\lim_{x\to t_r}\frac{nU'_{n-1}(x)(x-t_r)}{T_n(x)+(x-x_r)\,T'_n(x)}$$

$$=\frac{nU'_{n-1}(t_r)}{2T'_n(t_r)}=\frac{t_r}{2(1-t_r^2)},\tag{3.8}$$

and thereby

$$I(t_r)\approx\frac{1}{n}\sum_{k=1}^{n}{}'\frac{g(t_k)}{t_k-t_r}+\frac{1}{n}g'(t_r)+\frac{t_r\,g(t_r)}{2n(1-t_r^2)}\tag{3.9}$$

with the same estimate of remainder (3.6), which is also exact for $g(x)\in\pi_{2n}$.

Let $g^*(x)=g(x)(1-x^2)$, then $J(x,g)=I(x,g^*)$, so that from (1.7) we have

$$J(x)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)(1-t_k^2)}{t_k-x}+g(x)\frac{(1-x^2)\,U_{n-1}(x)}{T_n(x)},\qquad x\neq t_k,\tag{3.10}$$

and

$$J(x_j)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)(1-t_k^2)}{t_k-x_j},\qquad j=1,\dots,n-1.\tag{3.11}$$

From (3.10), we know that

$$H=H(g)=\frac{1}{\pi}\int_{-1}^{1}g(t)\sqrt{\frac{1+t}{1-t}}\,dt=-J(+1,g)\approx\frac{1}{n}\sum_{k=1}^{n}g(t_k)(1+t_k),\tag{3.12}$$

$$K=K(g)=\frac{1}{\pi}\int_{-1}^{1}g(t)\sqrt{\frac{1-t}{1+t}}\,dt=J(-1,g)\approx\frac{1}{n}\sum_{k=1}^{n}g(t_k)(1-t_k),\tag{3.12$'$}$$

since $T_n(\pm1)=(\pm1)^n$, $U_{n-1}(\pm1)=(\pm1)^{n-1}n$.

By (3.9), we also have

$$J(t_r)\approx\frac{1}{n}\sum_{k=1}^{n}{}'\frac{g(t_k)(1-t_k^2)}{t_k-t_r}+\frac{1}{n}g'(t_r)(1-t_r^2)-\frac{3}{2n}g(t_r).\tag{3.13}$$

From (3.6), we have the estimate of remainders of these formulas

$$|R_n[J(x)]|\leqslant\frac{1}{(2n+1)!\,2^{2n-1}}M_{2n+1}(g(x)(1-x^2)),\tag{3.14}$$

which shows they are exact for $g(x)\in\pi_{2n-2}$.

We may also write $H(x, g) = I(x, g(t)(1+t))$ and $K(x, g) = I(x, g(t)(1-t))$, so we have

$$H(x) \approx \frac{1}{n} \sum_{k=1}^{n} \frac{g(t_k)(1+t_k)}{t_k - x} + g(x) \frac{(1+x)\, U_{n-1}(x)}{T_n(x)}, \qquad -1 \leqslant x < 1,\ x \neq t_k, \tag{3.15}$$

$$K(x) \approx \frac{1}{n} \sum_{k=1}^{n} \frac{g(t_k)(1-t_k)}{t_k - x} + g(x) \frac{(1-x)\, U_{n-1}(x)}{T_n(x)}, \qquad -1 < x \leqslant 1,\ x \neq t_k; \tag{3.15$'$}$$

$$H(x_j) \approx \frac{1}{n} \sum_{k=1}^{n} \frac{g(t_k)(1+t_k)}{t_k - x_j}, \qquad j = 1, \ldots, n-1, \tag{3.16}$$

$$K(x_j) \approx \frac{1}{n} \sum_{k=1}^{n} \frac{g(t_k)(1-t_k)}{t_k - x_j}, \qquad j = 1, \ldots, n-1; \tag{3.16$'$}$$

$$H(t_r) \approx \frac{1}{n} {\sum_{k=1}^{n}}' \frac{g(t_k)(1+t_k)}{t_k - t_r} + \frac{1}{n} g'(t_r)(1+t_r) + \frac{g(t_r)}{2n} \frac{2-t_r}{1-t_r}, r = 1, \ldots, n, \tag{3.17}$$

$$K(t_r) \approx \frac{1}{n} {\sum_{n=1}^{n}}' \frac{g(t_k)(1-t_k)}{t_k - t_r} + \frac{1}{n} g'(t_r)(1-t_r) - \frac{g(t_r)}{2n} \frac{2+t_r}{1+t_r}, r = 1, \ldots, n, \tag{3.17$'$}$$

with the estimate of the remainders

$$|R_n[{}^{H}_{K}(x)]| \leqslant \frac{1}{(2n+1)!\, 2^{2n-1}} M_{2n+1}(g(t)(1 \pm t)), \tag{3.18}$$

which shows that they are exact for $g(t) \in \pi_{2n-1}$.

## 4. Formulas of Lobatto–Chebyshev Type

This method is based on the approximate formula [1]

$$J = \frac{1}{\pi} \int_{-1}^{1} f(x) \sqrt{1-x^2}\, dx \approx \frac{1}{n} \sum_{j=1}^{n-1} f(x_j)(1-x_j^2) \tag{4.1}$$

with remainder

$$R_n[J] = \frac{1}{(2n-2)!\, 2^{2n-1}} f^{(2n-2)}(\xi), \qquad -1 < \xi < 1. \tag{4.2}$$

Using (2.4), we may write

$$J(t)=\frac{1}{\pi}\int_{-1}^{1} G(x)\sqrt{1-x^2}\,dx-tg(t), \qquad G(x)=\frac{g(x)-g(t)}{x-t},$$

so that, from (4.1), we have

$$J(t)\approx\frac{1}{n}\sum_{j=1}^{n-1}\frac{g(x_j)(1-x_j^2)}{x_j-t}+\frac{g(t)}{n}\sum_{j=1}^{n-1}\frac{1-x_j^2}{t-x_j}-tg(t).$$

But

$$\begin{aligned}\sum_{j=1}^{n-1}\frac{1-x_j^2}{t-x_j}&=(1-t^2)\sum_{j=1}^{n-1}\frac{1}{t-x_j}+\sum_{j=1}^{n-1}(t+x_j)\\&=\frac{(1-t^2)\,U'_{n-1}(t)}{U_{n-1}(t)}+(n-1)t\end{aligned}$$

since $\sum_{j=1}^{n-1}x_j=0$ is the coefficient of $t^{n-2}$ in $U_{n-1}(t)$. Noting that

$$T_n(t)=tU_{n-1}(t)-U_{n-2}(t) \tag{4.3}$$

and thereby, by (3.7),

$$(1-t^2)\,U'_{n-1}(t)=tU_{n-1}(t)-nT_n(t), \tag{4.4}$$

we readily obtain (1.8).

For $t=x_s$, we have

$$\begin{aligned}J(x_s)\approx{}&\frac{1}{n}\sum_{j=1}^{n-1}{}'\,\frac{g(x_j)(1-x_j^2)}{x_j-x_s}+\frac{1}{n}\,g'(x_s)(1-x_s^2)\\&+\frac{g(x_s)}{n}\lim_{t\to x_s}\frac{(1-x_s)^2U_{n-1}(t)-nT_n(t)(x_s-t)}{(x_s-t)\,U_{n-1}(t)}.\end{aligned}$$

Proceeding as in (3.8) and using (3.5), (4.4), we may evaluate the involved limit which is equal to $-\frac{3}{2}t_s$. Hence

$$J(x_s)\approx\frac{1}{n}\sum_{j=1}^{n-1}{}'\,\frac{g(x_j)(1-x_j^2)}{x_j-x_s}+\frac{1}{n}\,g'(x_s)(1-x_s^2)-\frac{3}{2n}\,x_s\,g(x_s). \tag{4.5}$$

According to (4.2) and Lemma 1, the remainder of (1.8) or (4.5) is

$$R_n[J(t)]=\frac{1}{(2n-1)!\,2^{2n-1}}\,g^{(2n-1)}(\xi), \qquad -1<\xi<1. \tag{4.6}$$

In order to get formulas for $I(t)$, we first establish

$$I=\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,dx\approx\frac{1}{n}\sum_{j=0}^{n}\lambda_j f(x_j) \tag{4.7}$$

with estimate of the remainder

$$|R_n[I]|\leqslant\frac{1}{(2n-1)!\,2^{2n-1}}M_{2n}(f). \tag{4.8}$$

To prove this, we define

$$f(x)=F(x)(1-x^2)+Ax+B,\qquad A=\frac{f(1)-f(-1)}{2},\qquad B=\frac{f(1)+f(-1)}{2}, \tag{4.9}$$

or

$$F(x)=\frac{1}{2}\left[\frac{f(x)-f(-1)}{x+1}-\frac{f(x)-f(1)}{x-1}\right]. \tag{4.9$'$}$$

Then

$$I=\frac{1}{\pi}\int_{-1}^{1}F(x)\sqrt{1-x^2}\,dx+B.$$

By (4.1), we get

$$I\approx\frac{1}{n}\sum_{j=1}^{n-1}[f(x_j)-Ax_j-B]+B=\frac{1}{n}\sum_{j=1}^{n-1}f(x_j)+\frac{1}{n}B$$

which is just (4.7). From (4.2), we have, by Lemma 1,

$$\begin{aligned}R_n[I]&=\frac{1}{(2n-2)!\,2^{2n-1}}F^{(2n-2)}(\xi)\\&=\frac{1}{(2n-1)!\,2^{2n}}[f^{(2n-1)}(\xi_1)-f^{(2n-1)}(\xi_2)]\\&=\frac{\xi_1-\xi_2}{(2n-1)!\,2^{2n}}f^{(2n)}(\xi_3),\end{aligned}$$

which follows (4.8).

Now, from (1.1)$'$ and (4.7), we get

$$I(t)\approx\frac{1}{n}\sum_{j=0}^{n}\lambda_j\frac{g(x_j)}{x_j-t}-\frac{g(t)}{n}\sum_{j=0}^{n}\frac{\lambda_j}{x_j-t},\qquad t\neq x_j.$$

By (4.4), we see that

$$\sum_{j=0}^{n} \frac{\lambda_j}{x_j - t} = \frac{t}{1-t^2} - \sum_{j=1}^{n-1} \frac{1}{t - x_j} = \frac{t}{1-t^2} - \frac{U'_{n-1}(t)}{U_{n-1}(t)} = \frac{nT_n(t)}{(1-t^2)\,U_{n-1}(t)} \tag{4.10}$$

and therefore

$$I(t) \approx \frac{1}{n} \sum_{j=0}^{n} \lambda_j \frac{g(x_j)}{x_j - t} - g(t) \frac{T_n(t)}{(1-t^2)\,U_{n-1}(t)}, \qquad t \neq x_j, \tag{4.11}$$

of which (1.9) is a consequence.

For $t = x_s$, analogous to (3.9), we have

$$I(x_s) \approx \frac{1}{n} \sum_{j=0}^{n}{}' \lambda_j \frac{g(x_j)}{x_j - x_s} + \frac{1}{n} g'(x_s) + \frac{g(x_s)}{2n} \frac{x_s}{1 - x_s^2}, \qquad s = 1, \ldots, n-1, \tag{4.12}$$

where we have used the equality

$$\lim_{t \to x_s} \frac{(1-t^2)\,U_{n-1}(t) + nT_n(t)(t - x_s)}{(t - x_s)\,U_{n-1}(t)} = -\frac{x_s}{2}. \tag{4.13}$$

From (4.8), we have the estimate of the remainder of (4.11) or (4.12)

$$|R_n[I(t)]| \leqslant \frac{1}{2n+1} \cdot \frac{1}{(2n-1)!\, 2^{2n-1}} M_{2n+1}(g) < \frac{1}{(2n)!\ 2^{2n-1}} M_{2n+1}(g) \tag{4.14}$$

which shows they are exact for $g(x) \in \pi_{2n}$.

Similarly, by defining

$$f(t) = F(t)(1-t) + f(1) \qquad \text{or} \qquad F(t) = -\frac{f(t) - f(1)}{t - 1},$$

we may write

$$H = \frac{1}{\pi} \int_{-1}^{1} F(t) \sqrt{1 - t^2}\, dt + f(1).$$

Then, applying (4.1), (4.2) and Lemma 1, we may get

$$H \approx \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j f(x_j)(1 + x_j) \tag{4.15}$$

with remainder

$$R_n[H] = \frac{-1}{(2n-1)!\,2^{2n-1}} f^{(2n-1)}(\xi), \qquad -1 < \xi < 1. \tag{4.16}$$

In the same way, we may also get

$$K \approx \frac{1}{n} \sum_{j=1}^{n} \lambda_j f(x_j)(1-x_j) \tag{4.15$'$}$$

with remainder

$$R_n[K] = \frac{1}{(2n-1)!\,2^{2n-1}} f^{(2n-1)}(\xi), \qquad -1 < \xi < 1. \tag{4.16$'$}$$

Now, on account of (2.5), we may write

$$H(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{g(x)-g(t)}{x-t} \sqrt{\frac{1+x}{1-x}}\, dx + g(t)$$

and an analogous formula for $K(t)$. Then, by using (4.15)–(4.16)$'$ and Lemma 1, it is easy to obtain

$$H(t) \approx \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j \frac{g(x_j)(1+x_j)}{x_j - t} - g(t) \frac{T_n(t)}{(1-t)\,U_{n-1}(t)}, \qquad t \neq x_j, \tag{4.17}$$

$$K(t) \approx \frac{1}{n} \sum_{j=1}^{n} \lambda_j \frac{g(x_j)(1-x_j)}{x_j - t} - g(t) \frac{T_n(t)}{(1+t)\,U_{n-1}(t)}, \qquad t \neq x_j; \tag{4.17$'$}$$

$$H(t_k) \approx \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j \frac{g(x_j)(1+x_j)}{x_j - t_k}, \qquad k = 1,\ldots,n, \tag{4.18}$$

$$K(t_k) \approx \frac{1}{n} \sum_{j=1}^{n} \lambda_j \frac{g(x_j)(1-x_j)}{x_j - t_k}, \qquad k = 1,\ldots, n; \tag{4.18$'$}$$

$$H(x_s) \approx \frac{1}{n} \sum_{j=0}^{n-1}{}' \lambda_j \frac{g(x_j)(1+x_j)}{x_j - x_s} + \frac{1}{n} g'(x_s)(1+x_s) + \frac{g(x_s)}{2n} \frac{2-x_s}{1-x_s}, \tag{4.19}$$

$$K(x_s) \approx \frac{1}{n} \sum_{j=1}^{n}{}' \lambda_j \frac{g(x_j)(1-x_j)}{x_j - x_s} + \frac{1}{n} g'(x_s)(1-x_s) - \frac{g(x_s)}{2n} \frac{2+x_s}{1+x_s}, \tag{4.19$'$}$$

with remainders

$$R_n[{}^{H}_{K}(x)] = \frac{\mp 1}{(2n)!\,2^{2n-1}} g^{(2n)}(\xi), \qquad -1 < \xi < 1, \tag{4.20}$$

which show they are exact for $g(x) \in \pi_{2n-1}$.

*Remark.* Formulas for $H(t)$ and $K(t)$ may also be obtained even simpler from those for $I(t)$, but simple forms of remainders could not be obtained.

## 5. Formulas of Jacobi–Chebyshev Type

We may also start from the approximate formulas based on Jacobi polynomials, respectively, with respect to weights $\sqrt{(1+x)/(1-x)}$ and $\sqrt{(1-x)/(1+x)}$ [1]:

$$H = \frac{1}{\pi}\int_{-1}^{1} f(x)\sqrt{\frac{1+x}{1-x}}\,dx \approx \frac{2}{2n+1}\sum_{k=1}^{n} f(\tau_k)(1+\tau_k), \tag{5.1}$$

$$K = \frac{1}{\pi}\int_{-1}^{1} f(x)\sqrt{\frac{1-x}{1+x}}\,dx \approx \frac{2}{2n+1}\sum_{k=1}^{n} f(\sigma_k)(1-\sigma_k) \tag*{(5.1)$'$}$$

with remainders

$$R_n\begin{bmatrix} H \\ K \end{bmatrix} = \frac{1}{(2n)!\,2^{4n+1}} f^{(2n)}(\xi), \qquad -1<\xi<1, \tag{5.2}$$

where $\tau_k$, $\sigma_k$ are given by (1.13), being zeros of Jacobi polynomials

$$P_n(x) = P_n^{(-1/2,1/2)}(x), \qquad P_n(\tau_k) = 0, \tag{5.3}$$

$$Q_n(x) = P_n^{(1/2,-1/2)}(x), \qquad Q_n(\sigma_k) = 0, \tag*{(5.3)$'$}$$

$P_n(x)$ and $Q_n(x)$ are connected with Chebyshev polynomials by [1]

$$\left.\begin{aligned} P_n(2z^2-1) &= A_n T_{2n+1}(z)/z, \\ Q_n(2z^2-1) &= A_n U_{2n}(z). \end{aligned}\right\} \qquad (A_n = \text{const}) \tag{5.4}$$

Moreover, we also have

$$U_{2n}(x) = a_n P_n(x)\,Q_n(x) \qquad (a_n = \text{const}) \tag{5.5}$$

and

$$\tau_k = -\sigma_{n-k+1}, \qquad P_n(-x) = (-1)^n Q_n(x). \tag{5.6}$$

Then, as in Section 4, by using (5.1), we get

$$H(t) \approx \frac{1}{2n+1}\sum_{k=1}^{n} \frac{g(\tau_k)-g(t)}{\tau_k - t}(1+\tau_k) + g(t), \qquad t \neq \tau_k.$$

But from (5.4), we see that

$$P_n(t) + 2(1+t)\,P_n'(t) = (2n+1)\,Q_n(t), \tag{5.7}$$

so that

$$\sum_{k=1}^{n} \frac{1}{t-\tau_k} = \frac{P_n'(t)}{P_n(t)} = \frac{(2n+1)\,Q_n(t) - P_n(t)}{2(1+t)\,P_n(t)} \tag{5.8}$$

and

$$\sum_{k=1}^{n} \frac{1+\tau_k}{\tau_k - t} = \frac{2n+1}{2}\left[1 - \frac{Q_n(t)}{P_n(t)}\right].$$

Hence we obtain

$$H(t) \approx \frac{2}{2n+1} \sum_{k=1}^{n} \frac{g(\tau_k)(1+\tau_k)}{\tau_k - t} + g(t)\,\frac{Q_n(t)}{P_n(t)}, \qquad t \neq \tau_k. \tag{5.9}$$

Similarly, we also have

$$K(t) \approx \frac{2}{2n+1} \sum_{k=1}^{n} \frac{g(\sigma_k)(1-\sigma_k)}{\sigma_k - t} - g(t)\,\frac{P_n(t)}{Q_n(t)}, \qquad t \neq \sigma_k, \tag{5.9$'$}$$

which may also be obtained from (5.9) by

$$K(t, g(x)) = -H(-t, g(-x)), \tag{5.10}$$

(5.9) and (5.9)′ imply (1.12) and

$$K(\tau_k) \approx \frac{2}{2n+1} \sum_{j=1}^{n} \frac{g(\sigma_j)(1-\sigma_j)}{\sigma_j - \tau_k}, \qquad k = 1,\dots, n. \tag{1.12$'$}$$

Let $t \to \tau_s$ in (5.9), we get

$$H(\tau_s) \approx \frac{2}{2n+1} \sum_{k=1}^{n}{}' \frac{g(\tau_k)(1+\tau_k)}{\tau_k - \tau_s} + \frac{2}{2n+1}\, g'(\tau_s)(1+\tau_s)$$
$$+ \frac{g(\tau_s)}{2n+1} \lim_{t\to\tau_s} \left[\frac{(2n+1)\,Q_n(t)}{P_n(t)} + \frac{2(1+\tau_s)}{\tau_s - t}\right]. \tag{5.11}$$

We may evaluate the involved limit by (5.7), which equals

$$\frac{3}{2} + (1+\tau_s)\,\frac{Q_n'(\tau_s)}{Q_n(\tau_s)}.$$

But it is easy to verify

$$\frac{Q_n'(\tau_s)}{Q_n(\tau_s)}=\frac{U_{2n}'(z_s)}{4z_s U_{2n}(z_s)}=\frac{1}{4(1-z_s^2)}=\frac{1}{2(1-\tau_s)}, \qquad \tau_s=2z_s^2-1, \tag{5.12}$$

by virtue of (5.4). Hence we obtain

$$H(\tau_s)\approx\frac{2}{2n+1}\sum_{k=1}^{n}{}' \frac{g(\tau_k)(1+\tau_k)}{\tau_k-\tau_s}+\frac{2g'(\tau_s)}{2n+1}(1+\tau_s)$$
$$+\frac{g(\tau_s)}{2n+1}\frac{2-\tau_s}{1-\tau_s}, \qquad s=1,\dots,n. \tag{5.13}$$

Similarly, by (5.10),

$$K(\sigma_s)\approx\frac{2}{2n+1}\sum_{k=1}^{n}{}' \frac{g(\sigma_k)(1-\sigma_k)}{\sigma_k-\sigma_s}+\frac{2g'(\sigma_s)}{2n+1}(1-\sigma_s)$$
$$-\frac{g(\sigma_s)}{2n+1}\frac{2+\sigma_s}{1+\sigma_s}, \qquad s=1,\dots,n. \tag{5.13)'$$

By Lemma 1 and (5.2), the remainders of these formulas have the same form

$$R_n[{}^H_K(t)]=\frac{1}{(2n+1)!\,2^{4n+1}}\,g^{(2n+1)}(\xi), \qquad -1<\xi<1, \tag{5.14}$$

which shows they are exact for $g(x)\in\pi_{2n}$.

If we put

$$g(x)=(1+x)\,g_1(x)+g(-1),$$

we may write

$$I(x,g)=H(x,g_1)$$

on account of (2.3). Then, using (5.9) and (5.8), we obtain

$$I(x)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\tau_k)}{\tau_k-x}-\frac{g(-1)}{(2n+1)(1+x)}+g(x)\frac{Q_n(x)}{(1+x)\,P_n(x)},$$
$$x\neq\tau_k. \tag{5.15}$$

Similarly, since $I(x,g(t))=-I(-x,g(-t))$, we also have

$$I(x)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\sigma_k)}{\sigma_k-x}+\frac{g(1)}{(2n+1)(1-x)}-g(x)\frac{P_n(x)}{(1-x)\,Q_n(x)},$$
$$x\neq\sigma_k. \tag{5.15)'$$

Thus,

$$I(\sigma_j) \approx \frac{2}{2n+1} \sum_{k=1}^{n} \frac{g(\tau_k)}{\tau_k - \sigma_j} - \frac{g(-1)}{(2n+1)(1+\sigma_j)}, \qquad j = 1,\dots, n, \quad (5.16)$$

$$I(\tau_j) \approx \frac{2}{2n+1} \sum_{k=1}^{n} \frac{g(\sigma_k)}{\sigma_k - \tau_j} + \frac{g(1)}{(2n+1)(1-\tau_j)}, \qquad j = 1,\dots, n; \quad (5.16)'$$

and we may also get

$$I(\tau_s) \approx \frac{2}{2n+1} \sum_{k=1}^{n}{}' \frac{g(\tau_k)}{\tau_k - \tau_s} + \frac{2}{2n+1} g'(\tau_s) - \frac{g(-1)}{(2n+1)(1+\tau_s)} + \frac{g(\tau_s)}{2n+1} \frac{\tau_s}{1-\tau_s^2}, \quad (5.17)$$

$$I(\sigma_s) \approx \frac{2}{2n+1} \sum_{k=1}^{n}{}' \frac{g(\sigma_k)}{\sigma_k - \sigma_s} + \frac{2}{2n+1} g'(\sigma_s) + \frac{g(1)}{(2n+1)(1-\sigma_s)} + \frac{g(\sigma_s)}{2n+1} \frac{\sigma_s}{1-\sigma_s^2}. \quad (5.17)'$$

The remainders of these formulas have the same form

$$R_n[I(x)] = \frac{1}{(2n+2)!\, 2^{4n+1}} g^{(2n+2)}(\xi), \qquad -1 < \xi < 1, \quad (5.18)$$

which shows they are exact for $g(t) \in \pi_{2n+1}$.

Generally, these formulas for $I(x)$ are better than those given in Sections 3 or 4 in the sense that all of them are depending on the values of $g(x)$ at $n$ points.

Since

$$J(x, g(t)) = H(x, g(t)(1-t)) = K(x, g(t)(1+t)),$$

we may also have the following formulas:

$$J(x) \approx \frac{2}{2n+1} \sum_{k=1}^{n} \frac{g(\tau_k)(1-\tau_k^2)}{\tau_k - x} + g(x) \frac{(1-x)\, Q_n(x)}{P_n(x)}, \quad (5.19)$$

$$J(x) \approx \frac{2}{2n+1} \sum_{k=1}^{n} \frac{g(\sigma_k)(1-\sigma_k^2)}{\sigma_k - x} - g(x) \frac{(1+x)\, P_n(x)}{Q_n(x)}; \quad (5.19)'$$

$$J(\sigma_j) \approx \frac{2}{2n+1} \sum_{k=1}^{n} \frac{g(\tau_k)(1-\tau_k^2)}{\tau_k - \sigma_j}, \qquad j = 1,\dots, n, \quad (5.20)$$

$$J(\tau_j) \approx \frac{2}{2n+1} \sum_{k=1}^{n} \frac{g(\sigma_k)(1-\sigma_k^2)}{\sigma_k - \tau_j}, \qquad j = 1,\dots, n; \quad (5.20)'$$

$$J(\tau_s) \approx \frac{2}{2n+1} \sum_{k=1}^{n}{}' \frac{g(\tau_k)(1-\tau_k^2)}{\tau_k - \tau_s} + \frac{2g'(\tau_s)}{2n+1}(1-\tau_s^2) - \frac{3}{2n+1}\tau_s g(\tau_s),$$

$$s = 1,\ldots, n, \quad (5.21)$$

$$J(\sigma_s) \approx \frac{2}{2n+1} \sum_{k=1}^{n}{}' \frac{g(\sigma_k)(1-\sigma_k^2)}{\sigma_k - \sigma_s} + \frac{2g'(\sigma_s)}{2n+1}(1-\sigma_s^2) - \frac{3}{2n+1}\sigma_s g(\sigma_s),$$

$$s = 1,\ldots, n; \quad (5.21)'$$

with the estimates of the remainders

$$|R_n[J(x)]| \leqslant \frac{1}{(2n+1)!\, 2^{4n+1}} M_{2n+1}(g(t)(1 \mp t)), \tag{5.22}$$

which mean they are exact for $g(t) \in \pi_{2n-1}$.

## 6. Formulas of Simpson–Chebyshev Type

If the function $f(x)$ or $g(x)$ does not possess derivatives of sufficiently high order, then the approximate formulas derived based on tangential rule or trapezoidal rule in general are not as accurate as those derived based on Simpson's rule. Thus, in such cases, we may expect formulas for singular integrals based on the latter.

Analogous to Simpson's rule of quadrature for proper integrals, we easily get the Simpson–Chebyshev formula

$$I = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \approx \frac{1}{3n} \sum_{k=0}^{2n} \mu_k f(\xi_k), \tag{6.1}$$

where

$$\begin{gathered} \mu_0 = \mu_{2n} = \tfrac{1}{2}, \qquad \mu_{2m} = 1, \qquad m = 1,\ldots, n-1; \\ \mu_{2m+1} = 2, \qquad m = 0, 1,\ldots, n-1; \\ \xi_k = \cos(k/2n)\pi, \qquad k = 0, 1,\ldots, 2n \ (\xi_{2n-k} = -\xi_k), \end{gathered} \tag{6.2}$$

$\xi_1,\ldots, \xi_{2n-1}$ being the zeros of $U_{2n-1}(t)$. If $f(x) \in C^4[-1, 1]$, then we have the estimate of the remainder:

$$|R_{2n}[I]| \leqslant \frac{\pi^4}{2880 n^4} M_4(I(\cos\theta)), \qquad 0 \leqslant \theta \leqslant \pi, \tag{6.3}$$

which is obtained from the corresponding remainder formula for the integral [1]

$$I = \frac{1}{\pi} \int_0^{\pi} f(\cos\theta)\, d\theta.$$

Note that

$$U_{n-1}(\xi_{2m})=0 \qquad (\xi_{2m}=x_m), \qquad T_n(\xi_{2m-1})=0 \qquad (\xi_{2m-1}=t_m) \tag{6.4}$$

and

$$U_{2n-1}(t)=2U_{n-1}(t)\,T_n(t). \tag{6.5}$$

By (1.1)′, we have

$$\begin{aligned} I(t) &\approx \frac{1}{3n}\sum_{k=0}^{2n}\mu_k\frac{g(\xi_k)-g(t)}{\xi_k-t} \\ &= \frac{1}{3n}\sum_{k=0}^{2n}\mu_k\frac{g(\xi_k)}{\xi_k-t}+\frac{g(t)}{3n}\left(\sum_{m=1}^{n}\frac{1}{t-\xi_{2m-1}}+\sum_{k=0}^{2n}\frac{\lambda_k}{t-\xi_k}\right) \\ &= \frac{1}{3n}\sum_{k=0}^{2n}\mu_k\frac{g(\xi_k)}{\xi_k-t}+\frac{g(t)}{3}\left(\frac{U_{n-1}(t)}{T_n(t)}-\frac{2T_{2n}(t)}{(1-t^2)\,U_{2n-1}(t)}\right), \end{aligned}$$

because of (4.10). Since it is easy to prove

$$2(1-t^2)\,U_{n-1}^2(t)=1-T_{2n}(t),$$

we obtain

$$I(t)\approx\frac{1}{3n}\sum_{k=0}^{2n}\mu_k\frac{g(\xi_k)}{\xi_k-t}+\frac{g(t)}{3}\,\frac{1-3T_{2n}(t)}{(1-t^2)\,U_{2n-1}(t)}, \qquad t\neq\xi_k. \tag{6.6}$$

Thus, if we let

$$\left.\begin{aligned} \eta_{\pm j} &= \cos\frac{2j\pi\pm\operatorname{arc\,cos}\frac{1}{3}}{2n}, \qquad j=1,\dots,n-1, \\ \eta_{\pm n} &= \pm\cos\frac{\operatorname{arc\,cos}\frac{1}{3}}{2n}, \end{aligned}\right\} \tag{6.7}$$

then we have

$$I(\eta_j)\approx\frac{1}{3n}\sum_{k=0}^{2n}\mu_k\frac{g(\xi_k)}{\xi_k-\eta_j}, \qquad j=\pm1,\dots,\pm n. \tag{6.8}$$

Equation (6.8) may be used to solve singular integral equation (1.10) numerically in place of (1.8) as well, probably with better accuracy when $k(x,t)$ and $f(t)$ are not smooth enough.

We may get formulas for $I(\xi_r)$ also by letting $t \to \xi_r$ in (6.6). If $r$ is even, we have, by (4.13),

$$\begin{aligned} I(\xi_r) &\approx \frac{1}{3n} \sum_{k=0}^{2n}{}' \mu_k \frac{g(\xi_k)}{\xi_k - \xi_r} + \frac{\mu_r}{3n} g'(\xi_r) \\ &\quad + \frac{g(\xi_r)}{3n(1-\xi_r^2)} \lim_{t \to \xi_r} \frac{(1-t^2)\, U_{2n-1}(t) - 2n(\xi_r - t)\, T_{2n}(t)}{(\xi_r - t)\, U_{2n-1}(t)} \\ &= \frac{1}{3n} \sum_{k=0}^{2n}{}' \mu_k \frac{g(\xi_k)}{\xi_k - \xi_r} + \frac{\mu_r}{3n} g'(\xi_r) + \frac{g(\xi_r)}{6n} \frac{\xi_r}{1-\xi_r^2}; \end{aligned}$$

if $r$ is odd, we then have, by (3.8),

$$\begin{aligned} I(\xi_r) &\approx \frac{1}{3n} \sum_{k=0}^{2n}{}' \mu_k \frac{g(\xi_k)}{\xi_k - \xi_r} + \frac{\mu_r}{3n} g'(\xi_r) + \frac{g(\xi_r)}{6n} \frac{\xi_r}{1-\xi_r^2} \\ &\quad + \frac{g(\xi_r)}{3n} \lim_{t \to \xi_r} \frac{T_n(t) + n(\xi_r - t)\, U_{n-1}(t)}{(\xi_r - t)\, T_n(t)} \\ &= \frac{1}{3n} \sum_{k=0}^{2n}{}' \mu_k \frac{g(\xi_k)}{\xi_k - \xi_r} + \frac{\mu_r}{3n} g'(\xi_r) + \frac{g(\xi_r)}{3n} \frac{\xi_r}{1-\xi_r^2}. \end{aligned}$$

That is,

$$I(\xi_r) \approx \frac{1}{3n} \sum_{k=0}^{2n}{}' \mu_k \frac{g(\xi_k)}{\xi_k - \xi_r} + \frac{\mu_r}{3n} g'(\xi_r) + \frac{\mu_r g(\xi_r)}{6n} \frac{\xi_r}{1-\xi_r^2}, \qquad r = 1,\ldots, 2n-1. \tag{6.9}$$

Since $J(t, g) = I(t, g(x)(1-x^2))$, we may easily get

$$J(t) \approx \frac{1}{3n} \sum_{k=1}^{2n-1} \mu_k \frac{g(\xi_k)(1-\xi_k^2)}{\xi_k - t} + \frac{g(t)}{3} \frac{1 - 3T_{2n}(t)}{U_{2n-1}(t)}, \qquad t \neq \xi_k, \tag{6.10}$$

$$J(\eta_j) \approx \frac{1}{3n} \sum_{k=1}^{2n-1} \mu_k \frac{g(\xi_k)(1-\xi_k^2)}{\xi_k - \eta_j}, \qquad j = \pm 1,\ldots, \pm n, \tag{6.11}$$

$$J(\xi_r) \approx \frac{1}{3n} \sum_{k=1}^{2n-1}{}' \mu_k \frac{g(\xi_k)(1-\xi_k^2)}{\xi_k - \xi_r} + \frac{\mu_r}{3n} g'(\xi_r)(1-\xi_r^2) - \frac{\mu_r g(\xi_r)}{2n} \xi_r, \qquad r = 1,\ldots, 2n-1. \tag{6.12}$$

Since $H(t,g)=I(t,g(x)(1+x))$, $K(t,g)=I(t,g(x)(1-x))$, we also have

$$H(t)\approx\frac{1}{3n}\sum_{k=0}^{2n-1}\mu_k\frac{g(\xi_k)(1+\xi_k)}{\xi_k-t}+\frac{g(t)}{3}\frac{1-3T_{2n}(t)}{(1-t)\,U_{2n-1}(t)},\tag{6.13}$$

$$K(t)\approx\frac{1}{3n}\sum_{k=1}^{2n}\mu_k\frac{g(\xi_k)(1-\xi_k)}{\xi_k-t}+\frac{g(t)}{3}\frac{1-3T_{2n}(t)}{(1+t)\,U_{2n-1}(t)};\tag{6.13$'$}$$

$$H(\eta_j)\approx\frac{1}{3n}\sum_{k=0}^{2n-1}\mu_k\frac{g(\xi_k)(1+\xi_k)}{\xi_k-\eta_j},\qquad j=\pm 1,\dots,\pm n,\tag{6.14}$$

$$K(\eta_j)\approx\frac{1}{3n}\sum_{k=1}^{2n}\mu_k\frac{g(\xi_k)(1-\xi_k)}{\xi_k-\eta_j},\qquad j=\pm 1,\dots,\pm n;\tag{6.14$'$}$$

$$H(\xi_r)\approx\frac{1}{3n}\sum_{k=0}^{2n-1}{}'\mu_k\frac{g(\xi_k)(1+\xi_k)}{\xi_k-\xi_r}+\frac{\mu_r}{3n}g'(\xi_r)(1+\xi_r)+\frac{\mu_r g(\xi_r)}{6n}\frac{2-\xi_r}{1-\xi_r},$$
$$r=1,\dots,2n-1,\tag{6.15}$$

$$K(\xi_r)\approx\frac{1}{3n}\sum_{k=1}^{2n}{}'\mu_k\frac{g(\xi_k)(1-\xi_k)}{\xi_k-\xi_r}+\frac{\mu_r}{3n}g'(\xi_r)(1-\xi_r)-\frac{\mu_r g(\xi_r)}{6n}\frac{2+\xi_r}{1+\xi_r},$$
$$r=1,\dots,2n-1.\tag{6.15$'$}$$

Remainders of all the formulas obtained in this section may be estimated by means of (6.3) and Lemma 1, provided $g(x)\in C^5[-1,1]$.

Let us consider a numerical example as an illustration.

Consider the singular integral

$$I(t)=\frac{1}{\pi}\int_{-1}^{1}\frac{|x|}{(x-t)\sqrt{1-x^2}}\,dx=\frac{2t}{\pi\sqrt{1-t^2}}\ln\frac{1+\sqrt{1-t^2}}{|t|},$$
$$-1<t<1.$$

For $t=\frac{1}{2}$, we find

$$I(\tfrac{1}{2})=0.48405\dots$$

Here the density function $|x|$ has a corner point at $x=0$. The approximate values of $I(\frac{1}{2})$ have been calculated out by three of the methods described above. When the $n$'s are chosen to be odd for Lobatto–Chebyshev method and Jacobi–Chebyshev method and even for Simpson–Chebyshev method, better approximations are found, which are listed below. It is obvious that the last method is most effective in this example.

| $n$ | Lobatto–Chebyshev method | Jacobi–Chebyshev method | $n$ | Simpson–Chebyshev method |
|---|---|---|---|---|
| 3 | 0.44444 | 0.49240 | 2 | 0.49836 |
| 5 | 0.46667 | 0.48569 | | |
| 7 | 0.47446 | 0.48429 | 4 | 0.48602 |
| 9 | 0.47803 | 0.48392 | | |
| 11 | 0.47993 | 0.48383 | 6 | 0.48457 |
| 13 | 0.48106 | 0.48381 | | |
| 15 | 0.48179 | 0.48383 | 8 | 0.48424 |
| 17 | 0.48228 | 0.48385 | | |
| 19 | 0.48263 | 0.48387 | 10 | 0.48413 |
| 21 | 0.48288 | 0.48389 | | |
| 23 | 0.48307 | 0.48391 | 12 | 0.48409 |

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